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THE TWO DIMENSIONAL CONTACT PROBLEM OF A ROUGH STAMP SLIDING SLOWLY ON AN ELASTIC LAYER—II. THIN LAYER ASYMPTOTICS

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Abstract—An approximate solution is obtained for the contact problem of a layer of finite thickness loaded by a rough cylindrical stamp which moves along the boundary. The coefficient of friction is assumed to be constant. The lower side of the layer is attached to a rigid base. In the problem inertial forces are neglected and the solution is approximated by a plane strain solution. In this part of the paper the layer is assumed to be very thin with respect to the width of the contact region and the governing integral equation is reduced to one of the Wiener–Hopf type. In the solution terms of exponential decrease are neglected consistently.

1. INTRODUCTION

FOR a general statement of the problem to be dealt with, we refer to Part I of this investigation. The mathematical methods to be applied here differ considerably from those used for the thick layer asymptotics and this fact justifies a separate treatment. While the solution of the thick layer problem may be found by perturbing the half-plane solution, the perturbation originating from the lower plane boundary conditions, the thin layer asymptotics is based upon the fact that the effects of applied forces decrease exponentially from the point of application. Actually, a force acting on the boundary of a very thin layer has a local influence and we may use this fact for simplifying the governing integral equation. For more general considerations concerning thin layer asymptotics we refer to a recent paper [1].

2. THE THIN PLATE

In the sequel we shall retain the notation of Part I. The basic integral equation is equation (2.21) of Part I, viz.

$$\int_0^1 p(y)\left\{S_1(x-y)+\frac{f}{2(1-\nu)}S_2(x-y)\right\}\,\mathrm{d}y=\frac{(x-d)^2}{2R}+v_0,\qquad (0\le x\le 1),\tag{2.1}$$

which has to be solved subject to the condition

$$q\ll 1\,.\tag{2.2}$$

Here $S_1(t)$ and $S_2(t)$ are defined by [cf. Part I, (2.18), (2.19)]

$$S_1(t)=\int_{-\infty}^{\infty}\frac{K_1(w)}{w}\cos\frac{wt}{q}\,\mathrm{d}w,\tag{2.3}$$

$$S_2(t)=\int_{-\infty}^{\infty}\frac{K_2(w)}{w}\sin\frac{wt}{q}\,\mathrm{d}w,\tag{2.4}$$

with

$$w = q\xi, \tag{2.5}$$

while $K_1(w)$ and $K_2(w)$ are given in (2.15) and (2.16) of Part I. From (2.3) and (2.4) we derive the identity

$$S_1(t)+\frac{f}{2(1-\nu)}S_2(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-i(wt/q)}\left\{\frac{K_1(w)}{w}+\frac{if}{2(1-\nu)}\frac{K_2(w)}{w}\right\}\mathrm{d}w, \tag{2.6}$$

noting that the function $K_2(w)$ is even, while $K_1(w)$ is odd. In (2.6) the path of integration is along the real axis. The inversion formula of (2.6) is

$$\frac{K_1(w)}{w}+\frac{if}{2(1-\nu)}\frac{K_2(w)}{w} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{S_1(tq)+\frac{f}{2(1-\nu)}S_2(tq)\right\}\mathrm{e}^{itw}\,\mathrm{d}t, \tag{2.7}$$

which we write in the following form

$$\frac{K_1(qs)}{s}+\frac{if}{2(1-\nu)}\frac{K_2(qs)}{s} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{S_1(t)+\frac{f}{2(1-\nu)}S_2(t)\right\}\mathrm{e}^{ist}\,\mathrm{d}t. \tag{2.8}$$

To proceed we introduce the function

$$S(s) = \frac{K_1(qs)}{s}+\frac{if}{2(1-\nu)}\frac{K_2(qs)}{s}. \tag{2.9}$$

Similar to the treatment given in Ref. [1] we write the integral equation (2.1) as follows

$$\int_{-\infty}^{\infty} p(y)\left\{S_1(x-y)+\frac{f}{2(1-\nu)}S_2(x-y)\right\}\mathrm{d}y = v(x)+g(x), \qquad (-\infty < x < \infty), \tag{2.10}$$

where the functions $v(x)$ and $g(x)$ are defined by

$$v(x) = v_0+\frac{(x-d)^2}{2R}, \qquad (0 \le x \le 1), \tag{2.11}$$

$$v(x) = 0, \qquad (x < 0, x > 1), \tag{2.12}$$

$$g(x) = 0, \qquad (0 \le x \le 1), \tag{2.13}$$

$$g(x) \neq 0, \qquad (x < 0, x > 1), \tag{2.14}$$

and where, in addition

$$p = 0 \quad \text{for } x > 1, x < 0. \tag{2.15}$$

With the introduction of

$$P_-(s) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{0} p(1+y)\,\mathrm{e}^{isy}\,\mathrm{d}y, \tag{2.16}$$

$$G_+(s) = \frac{1}{\sqrt{(2\pi)}}\int_{0}^{\infty} g(1+y)\,\mathrm{e}^{isy}\,\mathrm{d}y, \tag{2.17}$$

$$G_-(s) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{0} g(y)\,\mathrm{e}^{isy}\,\mathrm{d}y, \tag{2.18}$$

the equation (2.10) may be transformed into

$$\sqrt{(2\pi)}P_-(s)S(s) = \sqrt{(2\pi)}G_+(s)+\sqrt{(2\pi)}G_-(s)+\int_{-\infty}^{\infty} v(1+x)\,e^{isx}\,dx, \tag{2.19}$$

where we have absorbed the factor 2π into p, so that

$$p(\text{Part II}) = 2\pi p(\text{Part I}) = \frac{1-\nu}{\mu}\text{ times the pressure.} \tag{2.20}$$

We note that $P_-(s)$ is regular in a lower half-plane $\operatorname{Im} s \leq -\varepsilon_1 < 0$, $G_-(s)$ is regular in a lower half-plane $\operatorname{Im} s \leq \varepsilon_2$ with $\varepsilon_2 > 0$, while $G_+(s)$ is regular in an upper half-plane $\operatorname{Im} s > -\varepsilon_3$ with $\varepsilon_3 > 0$. Obviously, the functional equation (2.19) holds in a strip below the real axis.

Equation (2.19) is completely equivalent with the integral equation (2.1) and it is valid for all values of the thickness parameter q. In order to obtain an asymptotic approximation for the case (2.2), we shall simplify (2.19) in the following way. We first calculate the integral in (2.19) and find

$$\begin{aligned}\int_{-1}^{0} v(1+x)\,e^{isx}\,dx &= \frac{1}{is}\left(v_0+\frac{(1-d)^2}{2R}\right)+\frac{1}{s^2}\,\frac{1-d}{R}-\frac{1}{iRS^3}\\ &\quad +e^{-is}\left[-\frac{1}{is}\left(v_0+\frac{d^2}{2R}\right)+\frac{d}{Rs^2}+\frac{1}{iRs^3}\right].\end{aligned} \tag{2.21}$$

In view of (2.2) the terms with e^{-is} in (2.21) and the function $G_-(s)$ contribute terms in the solution which are exponentially small. Therefore we drop them from (2.19) which becomes

$$\sqrt{(2\pi)}P_-(s)S(s) = \sqrt{(2\pi)}G_+(s)+\frac{1}{is}\left[v_0+\frac{(1-d)^2}{2R}\right]+\frac{1}{s^2}\,\frac{1-d}{R}-\frac{1}{iRs^3}. \tag{2.22}$$

This is the fundamental functional equation which can be solved with the aid of the standard Wiener–Hopf method.

3. SOLUTION OF THE FUNCTIONAL EQUATION (2.22)

For a general discussion of the problem (2.22) we refer to our Refs. [1] and [2] or to the literature on the Wiener–Hopf technique e.g. [3]. If we put s equal to zero in the function $S(s)$, given in (2.9), we obtain

$$S(0) = \frac{q(1-2\nu)}{2(1-\nu)^2} \tag{3.1}$$

and its asymptotic formula for $|s| \to \infty$ in the strip is given by

$$S(\infty) \sim -\frac{1}{s}\left[1+\frac{if(1-2\nu)}{2(1-\nu)}\right]. \tag{3.2}$$

We decompose $S(s)$ according to

$$S(s) = \frac{S_-(s)}{S_+(s)}, \tag{3.3}$$

where $S_-(s)$ is regular in a lower half-plane and $S_+(s)$ is regular in an upper half-plane. The two half-planes have a strip below the real axis in common. Using (3.3) we write (2.22) in the form

$$s^3S_-(s)P_-(s) = S_+(s)\left[\frac{1}{\sqrt{(2\pi)}}\left\{\frac{s^2}{i}\left(v_0+\frac{(1-d)^2}{R}\right)+\frac{1-d}{R}s-\frac{1}{iR}\right\}+s^3G_+(s)\right], \tag{3.4}$$

and we conclude that both sides of this equation must be equal to a polynomial.

From the asymptotic formulae of $S_-(s)$ and $S_+(s)$ in their respective half-planes as $|s| \to \infty$ (cf. Appendix)

$$S_-(s) \sim 0(s^{-\theta}), \tag{3.5}$$

$$S_+(s) \sim 0(s^{1-\theta}), \tag{3.6}$$

and from the assumption of bounded pressure, we see that the degree of the polynomial does not exceed one. We then find

$$P_-(s) = \frac{c_0+c_1s}{s^3S_-(s)}, \tag{3.7}$$

$$G_+(s) = \frac{c_0+c_1s}{s^3S_+(s)}+\frac{1}{\sqrt{(2\pi)}}\left\{\frac{i}{s}\left(v_0+\frac{(1-d)^2}{2R}\right)-\frac{1-d}{R}\frac{1}{s^2}+\frac{1}{iRs^3}\right\}. \tag{3.8}$$

The solution (3.7), (3.8) contains four unknown constants: c_0, c_1, v_0 and d, which have to comply with the conditions expressing the regularity of $G_+(s)$ at $s = 0$, yielding

$$c_0 = -\frac{1}{\sqrt{(2\pi)}}\frac{S_+(0)}{iR}, \tag{3.9}$$

$$c_1 = \frac{1}{\sqrt{(2\pi)}}\frac{1}{R}[(1-d)S_+(0)+iS'_+(0)], \tag{3.10}$$

$$\frac{i}{2}S''_+(0)+(1-d)S'_+(0)-iS_+(0)\left[v_0R+\frac{(1-d)^2}{2}\right] = 0. \tag{3.11}$$

The equations (3.9)–(3.11) constitute three equations for the four unknowns. By considering the system with a reversed motion:

$$f \to -f \tag{3.12}$$

we can supply the additional equation.

Denoting the functions pertaining to the reversed motion by a star

$$S(s) \to S^*(s), \qquad c_0 \to c_0^*, \qquad c_1 \to c_1^*, \tag{3.13}$$

we obviously have

$$v_0^* = v_0, \tag{3.14}$$

$$d^* = 1-d. \tag{3.15}$$

Analogous to (3.9)–(3.11) we then obtain

$$c_0^* = -\frac{1}{\sqrt{(2\pi)}}\frac{S_+^*(0)}{iR}, \tag{3.16}$$

$$c_1^* = \frac{1}{\sqrt{(2\pi)}}\frac{1}{R}[\mathrm{d}S_+^*(0)+iS_+^{*\prime}(0)], \tag{3.17}$$

$$\frac{i}{2}S_+^{*\prime\prime}(0)+dS_+^{*\prime}(0)-iS_+^*(0)\left(v_0R+\frac{d^2}{2}\right) = 0. \tag{3.18}$$

The constants c_0, c_1, c_0^*, c_1^*, v_0 and d may be determined by the equations (3.9)–(3.11) and (3.16)–(3.18).

4. THE PRESSURE AND THE DISPLACEMENT

From (3.7) and (2.16) we find for the pressure p

$$p(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty+i\gamma}^{\infty+i\gamma}\frac{c_0+c_1s}{s^3S_-(s)}\mathrm{e}^{-is(x-1)}\,\mathrm{d}s, \qquad (x > \tfrac{1}{2}), \tag{4.1}$$

where $-\varepsilon_3 < \gamma < -\varepsilon_1$.

Referring to (2.15) we see from (4.1) that $p(x)$ satisfies

$$p(x) = 0, \qquad (x \geq 1). \tag{4.2}$$

From (3.8) and (2.17) we derive

$$g(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty+i\gamma}^{\infty+i\gamma}\left[\frac{c_0+c_1s}{s^3S_+(s)}+\frac{1}{\sqrt{(2\pi)}}\left\{\frac{i}{s}\left(v_0+\frac{(1-d)^2}{2R}\right)-\frac{1-d}{Rs^2}+\frac{1}{iRs^3}\right\}\right]\mathrm{e}^{-is(x-1)}\,\mathrm{d}s,$$

$$(x > 0). \tag{4.3}$$

From the representation (4.3) it may be shown that $g(x)$ satisfies (2.13), while $g(1)$ and $g'(1)$ are given by

$$g(1) = v_0+\frac{(1-d)^2}{2R}, \tag{4.4}$$

and

$$g'(1) = \frac{1-d}{R}, \tag{4.5}$$

respectively.

In order to find the pressure $p(x)$ for $x < \frac{1}{2}$ and the displacement $g(x)$ for $x < 0$, we use the results obtained for the reversed motion (3.12). In this way we arrive at

$$p(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty+i\gamma}^{\infty+i\gamma}\frac{c_0^*+c_1^*s}{s^3S_-^*(s)}\mathrm{e}^{isx}\,\mathrm{d}s, \qquad (x < \tfrac{1}{2}), \tag{4.6}$$

and

$$g(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty+i\gamma}^{\infty+i\gamma} \left[\frac{c_0^* + c_1^* s}{s^3 S_+^*(s)} + \frac{1}{\sqrt{(2\pi)}} \left\{ \frac{i}{s}\left(v_0 + \frac{d^2}{2R} \right) - \frac{d}{Rs^2} + \frac{1}{iRs^3} \right\} \right] e^{isx}\, ds, \qquad (x < 1). \tag{4.7}$$

From these results we infer some conclusions similar to (4.2), (4.4) and (4.5).

The total force P is found by integrating $p(x)$ over the interval $0 \leq x \leq 1$. We find

$$P = -\frac{i}{\sqrt{(2\pi)}} \int_{-\infty+i\gamma}^{\infty+i\gamma} \left\{ \frac{c_0 + c_1 s}{s^4 S_-(s)} + \frac{c_0^* + c_1^* s}{s^4 S_-^*(s)} \right\} e^{is/2}\, ds. \tag{4.8}$$

5. NUMERICAL RESULTS

Using the Tables 1 and 2 (cf. Appendix) we have evaluated the displacement v_0 for some values of Poisson's ratio and the friction coefficient. The results are given in Table 3. We also obtained some values for the constant d which are given in Table 4. By contour integration of (4.8) we calculated the total force P. We found

$$P = \sqrt{(2\pi)} \left[\frac{c_0}{S_-(0)} a + \frac{c_1}{S_-(0)} b + \frac{c_0^*}{S_-^*(0)} a^* + \frac{c_1^*}{S_-^*(0)} b^* \right], \tag{5.1}$$

where

$$a = -\frac{i}{48} + \frac{1}{8} \frac{S'_-(0)}{S_-(0)} + \frac{i}{2} \left(\frac{S'^2_-(0)}{S^2_-(0)} - \frac{1}{2} \frac{S''_-(0)}{S_-(0)} \right) + \left(-\frac{1}{6} \frac{S'''_-(0)}{S_-(0)} + \frac{S'_-(0) S''_-(0)}{S^2_-(0)} - \frac{S'^3_-(0)}{S^3_-(0)} \right), \tag{5.2}$$

$$b = -\frac{1}{8} - \frac{i}{2} \frac{S'_-(0)}{S_-(0)} + \left(\frac{S'^2_-(0)}{S^2_-(0)} - \frac{1}{2} \frac{S''_-(0)}{S_-(0)} \right), \tag{5.3}$$

while a^* and b^* are similar expressions, formed from $S_-^*(0)$ and the derivatives.

TABLE 1

ν	f	$S_+(0)$	$S'_+(0)$	$S''_+(0)$	$S'''_+(0)$	$S_-(0)$	$S'_-(0)$	$S''_-(0)$	$S'''_-(0)$
0·2	0·1	$-2{\cdot}1333\, q^{-1}$	$+0{\cdot}39\, i$	$+0{\cdot}011\, q$	$+0{\cdot}096\, iq^2$	1	$-0{\cdot}16\, iq$	$-0{\cdot}05\, q^2$	$-0{\cdot}003\, iq^3$
0·2	0·5	$-2{\cdot}1333\, q^{-1}$	$+0{\cdot}45\, i$	$+0{\cdot}028\, q$	$+0{\cdot}16\, iq^2$	1	$-0{\cdot}12\, iq$	$-0{\cdot}03\, q^2$	$+0{\cdot}017\, iq^3$
0·4	0·1	$-3{\cdot}6\, q^{-1}$	$-0{\cdot}69\, i$	$+3{\cdot}46\, q$	$+15{\cdot}95\, iq^2$	1	$+0{\cdot}05\, iq$	$+0{\cdot}43\, q^2$	$-1{\cdot}61\, iq^3$
0·4	0·5	$-3{\cdot}6\, q^{-1}$	$-1{\cdot}91\, i$	$+8{\cdot}98\, q$	$+51{\cdot}86\, iq^2$	1	$-0{\cdot}21\, iq$	$-0{\cdot}37\, q^2$	$+1{\cdot}41\, iq^3$

TABLE 2

ν	f	$S_+^*(0)$	$S_+^{*\prime}(0)$	$S_+^{*\prime\prime}(0)$	$S_+^{*\prime\prime\prime}(0)$	$S_-^*(0)$	$S_-^{*\prime}(0)$	$S_+^{*\prime\prime}(0)$	$S_-^{*\prime\prime\prime}(0)$
0·2	0·1	1	$-0{\cdot}16\, iq$	$+0{\cdot}105\, q^2$	$-0{\cdot}08\, iq^3$	$-0{\cdot}46875\, q$	$+0{\cdot}09\, iq^2$	$-0{\cdot}034\, q^3$	$+0{\cdot}04\, iq^4$
0·2	0·5	1	$-0{\cdot}12\, iq$	$+0{\cdot}063\, q^2$	$-0{\cdot}018\, iq^3$	$-0{\cdot}46875\, q$	$+0{\cdot}10\, iq^2$	$-0{\cdot}049\, q^3$	$+0{\cdot}07\, iq^4$
0·4	0·1	1	$+0{\cdot}05\, iq$	$-0{\cdot}42\, q^2$	$-1{\cdot}73\, iq^3$	$-0{\cdot}2778\, q$	$-0{\cdot}15\, iq^2$	$-0{\cdot}29\, q^3$	$+0{\cdot}92\, iq^4$
0·4	0·5	1	$-0{\cdot}21\, iq$	$+0{\cdot}46\, q^2$	$+0{\cdot}88\, iq^3$	$-0{\cdot}2778\, q$	$-0{\cdot}42\, iq^2$	$-0{\cdot}85\, q^3$	$+1{\cdot}55\, iq^4$

TABLE 3

ν	f	v_0
0·2	0·1	$\frac{1}{R}[-0{\cdot}125-0{\cdot}086\,q+0{\cdot}025\,q^2+0(q^3)]$
0·2	0·5	$\frac{1}{R}[-0{\cdot}125-0{\cdot}083\,q+0{\cdot}014\,q^2+0(q^3)]$
0·4	0·1	$\frac{1}{R}[-0{\cdot}125+0{\cdot}060\,q-0{\cdot}475\,q^2+0(q^3)]$
0·4	0·5	$\frac{1}{R}[-0{\cdot}125+0{\cdot}014\,q-0{\cdot}480\,q^2+0(q^3)]$

TABLE 4

ν	f	d
0·2	0·1	$0{\cdot}5+0{\cdot}011\,q+0{\cdot}051\,q^2+0(q^3)$
0·2	0·5	$0{\cdot}5+0{\cdot}046\,q+0{\cdot}023\,q^2+0(q^3)$
0·4	0·1	$0{\cdot}5-0{\cdot}071\,q+0{\cdot}453\,q^2+0(q^3)$
0·4	0·5	$0{\cdot}5-0{\cdot}238\,q+1.464\,q^2+0(q^3)$

In (5.1) the contributions resulting from the poles of $S_-(s)$ and $S_-^*(s)$ have been omitted in view of the consistency with the approximations, made *a priori*. In Table 5 we show the results of this computation.

The pressure distribution $p(x)$ has been calculated by contour integration of (4.1) and (4.6). In the region $\frac{1}{2} \leq x < 1-\varepsilon_4$, with $\varepsilon_4 > q/a$ and in the region $\varepsilon_5 < x \leq \frac{1}{2}$ with $\varepsilon_5 > q/a$ we may neglect the poles of $S_-(s)$ and $S_-^*(s)$, respectively, as they contribute terms which are exponentially small. In the first region we find

$$p(x) = \sqrt{(2\pi)}\left[\frac{c_0}{S_-(0)}\left\{i\frac{S_-'^2(0)}{S_-^2(0)}-\frac{1}{2}i\frac{S_-''(0)}{S_-(0)}-(x-1)\frac{S_-'(0)}{S_-(0)}-\frac{i}{2}(x-1)^2\right\}\right.$$
$$\left.+\frac{c_1}{S_-(0)}\left\{-i\frac{S_-'(0)}{S_-(0)}+(x-1)\right\}\right], \qquad (\tfrac{1}{2} \leq x < 1-\varepsilon_4), \tag{5.4}$$

TABLE 5

ν	f	P
0·2	0·1	$\frac{1}{R}[0{\cdot}178\,q^{-1}+0{\cdot}024+0{\cdot}027\,q-0{\cdot}037\,q^2+0(q^3)]$
0·2	0·5	$\frac{1}{R}[0{\cdot}178\,q^{-1}+0{\cdot}065+0{\cdot}005\,q+0{\cdot}020\,q^2+0(q^3)]$
0·4	0·1	$\frac{1}{R}[0{\cdot}3\,q^{-1}-0{\cdot}202+0{\cdot}094\,q+3{\cdot}923\,q^2+0(q^3)]$
0·4	0·5	$\frac{1}{R}[0{\cdot}3\,q^{-1}-0{\cdot}253+3{\cdot}119\,q+26{\cdot}471\,q^2+0(q^3)]$

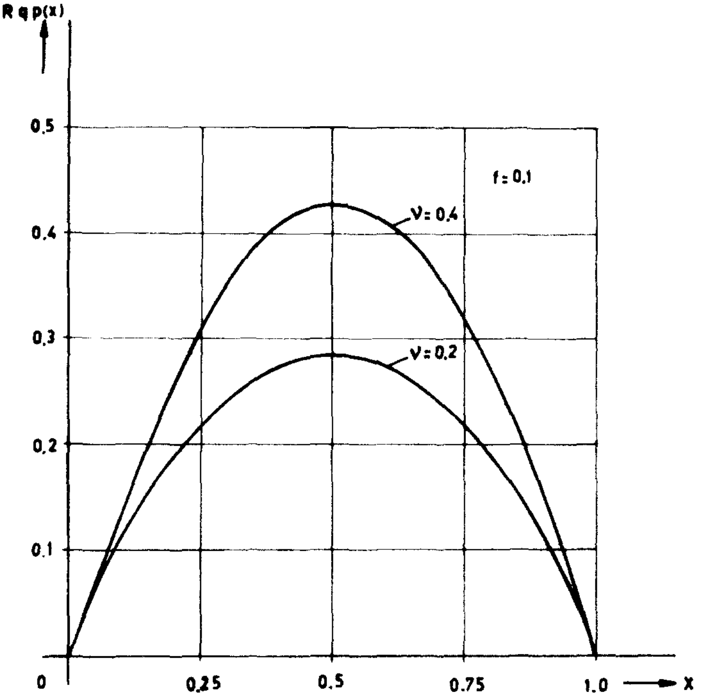

FIG. 1. The pressure distribution for $f = 0, 1$.

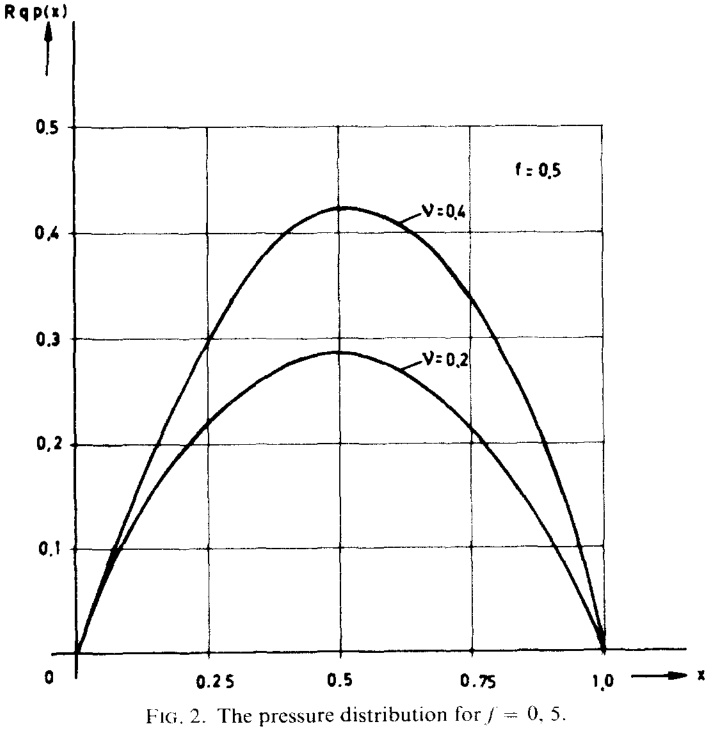

FIG. 2. The pressure distribution for $f = 0, 5$.

and a similar expression is obtained for the region $\frac{1}{2} \geq x > \varepsilon_5$. In the neighbourhood of $x = 1$, the pressure $p(x)$ tends to zero as

$$p(x) = 0[(1-x)^{1-\theta}], \qquad x \approx 1, \tag{5.5}$$

while near $x = 0$ the corresponding formula is

$$p(x) = 0[x^{\theta}], \qquad x \approx 0. \tag{5.6}$$

In Figs. 1 and 2 $Rq\, p(x)$ is given as a function of x for some values of Poisson's ratio and the friction constant.

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APPENDIX

The problem of the decomposition of the function $S(s)$, given in (2.9) is solved by a straightforward splitting of the product representation. To facilitate the discussion we introduce

$$z = 2qs, \qquad t = \frac{f}{4(1-\nu)}, \qquad \alpha = 3-4\nu, \qquad \beta = 2(1-2\nu), \qquad S(s) = q\Psi(z), \tag{A.1}$$

with

$$\Psi(0) = -\frac{1-2\nu}{2(1-\nu)^2}, \tag{A.2}$$

$$\Psi'(0) = -if\frac{1-4\nu}{8(1-\nu)^2}. \tag{A.3}$$

Writing

$$\Psi(z) = \frac{T(z)}{N(z)}, \tag{A.4}$$

with

$$T(z) = 2\left(1-\alpha\frac{\sinh z}{z}\right)+2itz\left[1-\alpha\beta\left(\frac{\cosh z-1}{z^2}\right)\right], \tag{A.5}$$

$$N(z) = \alpha\cosh z+\tfrac{1}{2}z^2+(\alpha+\tfrac{1}{2}\beta^2), \tag{A.6}$$

we see that $\Psi(z)$ is a meromorphic function. The zero's of $N(z)$ are

(a) two points on the imaginary axis: ia and $-ia$, with $a > 0$;

(b) an infinite sequence of points in the complex plane: $p_k, -p_k, \bar{p}_k, -\bar{p}_k, (k = 1, 2, \ldots)$, with $\operatorname{Im} p_k > 0$, $\operatorname{Re} p_k > 0$.

The zero's of $T(z)$ are, for $t > 0$,

(a) one point on the imaginary axis: ic, with $c < 0$;

(b) an even number of points on the same axis: $ib_1, ib_2 \ldots ib_{2n}$, with $b_k > 0$. It appears that $n = 0, 1$ or 2;

(c) an infinite sequence of complex numbers $q_k, -\bar{q}_k, (k = 1, 2, \ldots)$, with $\operatorname{Im} q_k > 0$, $\operatorname{Re} q_k > 0$;

(d) an infinite sequence of complex numbers $r_k, -\bar{r}_k, (k = 1, 2, \ldots)$, with $\operatorname{Im} r_k < 0$, $\operatorname{Re} r_k > 0$.

We have derived some asymptotic expressions for the roots. We found

$$\left.\begin{aligned} p_k &= \log\left(\frac{4k^2\pi^2}{\alpha}\right) + 2k\pi i + 0\left(\frac{\log k}{k}\right), \\ q_k &= \log\left[\frac{8t\pi^2}{\alpha\sqrt{(1+\beta^2t^2)}}k^2\right] + i\left(2k\pi - \frac{\pi}{2} - \varphi_0\right) + 0\left(\frac{\log k}{k}\right), \\ r_k &= \log\left[\frac{8t\pi^2}{\alpha\sqrt{(1+\beta^2t^2)}}k^2\right] - i\left(2k\pi + \frac{\pi}{2} + \varphi_0\right) + 0\left(\frac{\log k}{k}\right), \end{aligned}\right\} k \gg 1 \tag{A.7}$$

with

$$\varphi_0 = \arctan \beta t, \qquad 0 < \varphi_0 < \frac{\pi}{2}. \tag{A.8}$$

According to a well-known theorem we may represent $\Psi(z)$ in the following form

$$\begin{aligned} \Psi(z) = \Psi(0)\, e^{z\frac{\Psi'(0)}{\Psi(0)}} &\cdot \frac{(1-z/ib_1)(1-z/ib_2)\ldots(1-z/ic)}{(1-z/ia)(1+z/ia)} \cdot \\ &\prod_{k=1}^{\infty} \frac{(1-z/q_k)(1+z/\bar{q}_k)(1-z/r_k)(1+z/\bar{r}_k)}{(1-z/p_k)(1+z/p_k)(1-z/\bar{p}_k)(1+z/\bar{p}_k)} \\ &\exp z\left\{\left(\frac{1}{ib_1}+\frac{1}{ib_2}+\ldots+\frac{1}{ic}\right)+\left(\frac{1}{q_k}-\frac{1}{\bar{q}_k}+\frac{1}{r_k}-\frac{1}{\bar{r}_k}\right)\right\}. \end{aligned} \tag{A.9}$$

The representation (A.9) is

$$\Psi = \Psi(0)\, e^{\text{const.}z}\frac{P_1(z)}{P_2(z)}, \tag{A.10}$$

where P_1 and P_2 denote the products in nominator and denominator of (A.9) respectively.

It may be shown that (A.10) may be written as

$$\begin{aligned} \Psi(z) = \Psi(0)&\frac{(1-z/ib_1)(1-z/ib_2)\ldots(1-z/ic)}{(1-z/ia)(1+z/ia)} \cdot \\ &\prod_{k=1}^{\infty} \frac{(1-z/q_k)(1+z/\bar{q}_k)(1-z/r_k)(1+z/\bar{r}_k)}{(1-z/p_k)(1+z/p_k)(1-z/\bar{p}_k)(1+z/\bar{p}_k)}. \end{aligned} \tag{A.11}$$

From (A.11) we derive for the decomposition

$$\Psi = \frac{\Psi_-(z)}{\Psi_+(z)} \tag{A.12}$$

the following representations

$$\Psi_+(z) = \frac{(1+z/ia)\,.\,e^{\chi(z)}}{\Psi(0)(1-z/ic)} \prod_{k=1}^{\infty} \frac{(1+z/p_k)(1-z/\bar{p}_k)}{(1-z/r_k)(1+z/\bar{r}_k)}, \tag{A.13}$$

$$\Psi_-(z) = \frac{(1-z/ib_1)(1-z/ib_2)\ldots e^{\chi(z)}}{(1-z/ia)} \prod_{k=1}^{\infty} \frac{(1-z/q_k)(1+z/\bar{q}_k)}{(1-z/p_k)(1+z/\bar{p}_k)}, \tag{A.14}$$

where $\chi(z)$ has to be determined.

With the aid of (A.7) it may easily be proved that the products in (A.13) and (A.14) converge. If we require algebraic behaviour for $\Psi_+(z)$ and $\Psi_-(z)$ in their respective half-planes, as $|z| \to \infty$, we have to take

$$\chi(z) = 0. \tag{A.15}$$

This may be proved as follows.

We compare the product $P(z)$, defined by

$$P(z) = \prod_{k=1}^{\infty} (1+z/p_k)\,e^{-z/2k\pi i}\,.\prod_{k=1}^{\infty} (1-z/\bar{p}_k)\,e^{-z/2k\pi i},$$

with

$$J(z) = \prod_{k=1}^{\infty} (1+z/2k\pi i)^2\,e^{-z/k\pi i} = e^{-\gamma z/\pi i}\left[\frac{1}{\Gamma(1+z/2\pi i)}\right]^2, \tag{A.16}$$

where γ is Euler's constant. Then we find

$$\frac{P(z)}{J(z)} \sim 0(1), \quad \text{for } |z| \to \infty \text{ in the upper half-plane.} \tag{A.17}$$

Writing the product in (A.13) in the following form

$$\frac{\prod_{k=1}^{\infty} (1+z/p_k)\,e^{-z/2k\pi i}\,.\prod_{k=1}^{\infty} (1-z/\bar{p}_k)\,e^{-z/2k\pi i}}{\prod_{k=1}^{\infty} (1-z/r_k)\,e^{-z/2k\pi i}\,.\prod_{k=1}^{\infty} (1+z/\bar{r}_k)\,e^{-z/2k\pi i}},$$

we see that for $|z| \to \infty$ its asymptotic behaviour in the upper half-plane is given by

$$\begin{aligned} &0(1)\left[e^{-\gamma z/2\pi i}\frac{1}{\Gamma(1+z/2\pi i)}\right]^2\left[e^{\gamma z/2\pi i}\frac{\Gamma\left(1+\dfrac{\pi/2+\varphi_0}{2\pi}+\dfrac{z}{2\pi i}\right)}{\Gamma\left(1+\dfrac{\pi/2+\varphi_0}{2\pi}\right)}\right]^2 \\ &= 0(1)\left(\frac{\Gamma(5/4+\varphi_0/2\pi+z/2\pi i)}{\Gamma(1+z/2\pi i)}\right)^2 \sim z^{1/2+\varphi_0/\pi} \end{aligned} \tag{A.18}$$

From (A.18) we see that we must take $\chi = 0$, as has already been stated by (A.15). For $\nu \neq \frac{1}{2}$, $c \neq 0$ and we conclude that the asymptotic formula for $S_+(s)$ is

$$S_+(s) \sim 0(s^{1-\theta}) \tag{A.19}$$

conform to (3.6), as

$$\varphi_0/\pi = \tfrac{1}{2} - \theta \tag{A.20}$$

From (A.19) we infer that

$$S_-(s) \sim 0(s^{-\theta}), \tag{A.21}$$

according to (3.5).

Numerical values of $S'_+(0)$, $S''_+(0)$, $S'''_+(0)$ and $S'_-(0)$, $S''_-(0)$, $S'''_-(0)$ are given in Table 1. For the reversed motion (3.12) we define

$$\Psi^* = \frac{T^*(z)}{N(z)}, \tag{A.22}$$

which satisfies

$$\Psi^*(-z) = \Psi(z) = \frac{T^*(-z)}{N(z)} = \frac{T(z)}{N(z)}. \tag{A.23}$$

From (A.23) we infer that

$$\Psi^*_-(z) = \frac{1}{\Psi_+(-z)}, \tag{A.24}$$

$$\Psi^*_+(z) = \frac{1}{\Psi_-(-z)}. \tag{A.25}$$

From (A.24), (A.25) we have found the numerical values of $S^{*\prime}_+(0)$, $S^{*\prime\prime}_+(0)$, $S^{*\prime\prime\prime}_-(0)$ and $S^{*\prime}_-(0)$, $S^{*\prime\prime}_-(0)$, $S^{*\prime\prime\prime}_-(0)$, which are presented in Table 2.

We note that we have

$$\Psi_-(0) = 1; \qquad \Psi_+(0) = \frac{1}{\Psi(0)} = -\frac{2(1-\nu)^2}{1-2\nu}. \tag{A.26}$$

In our numerical calculations it is preferable to normalize $S_-(0)$ and $S_+(0)$ in another way. We take

$$S_-(0) = 1, \qquad S_+(0) = \frac{1}{S(0)} = -\frac{2(1-\nu)^2}{q(1-2\nu)}. \tag{A.27}$$

From (A.24) and (A.25) we see that then

$$S^*_+(0) = 1, \qquad S^*_-(0) = -\frac{q(1-2\nu)}{2(1-\nu)^2}. \tag{A.28}$$

From the definition $S_-(s) = S(s)S_+(s)$ we further derive that

$$-S'_-(0) = S'(0)\frac{2(1-\nu)^2}{q(1-2\nu)} + \frac{q(1-2\nu)}{2(1-\nu)^2}S'_+(0), \tag{A.29}$$

where

$$S'(0) = -2ifq^2 \frac{1-4\nu}{8(1-\nu)^2}. \tag{A.30}$$

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Абстракт—Получается приближенное решение контактной задачи, касающейся слоя конечной толщины, нагруженного шереховатымцилиндрическим щтомпом, который движется вдоль границы. Предлагается постоянный козффициент трения. Нижняя сторона слоя прикреплена к жесткомм основанию. В задаче пренебрегается инерционными условиями. Решение приближается с помощбю решения для плоской деформации. В зтой части работы предполагается, по отношению к ширине района контакта, очень тонний слой. Определяющее интегральное уравнение сводотся к уравнению типа Винера-Хопфа. В решении пренебрегаются постепенно членами зкпотенцияльного убывания.